CURVE-TRACING

13.1. INTRODUCTION

In this chapter we shall deal with the graph of those curves which are useful for the subsequent chapters on "area of the curves", 'length of the arc' etc. The main purpose of this chapter is to point out those rules which are used in tracing the graph of a curve. At the outset, we shall describe the main rules of curve-tracing and then afterwards we shall use them in tracing the graph of aforesaid curves. First of all, we shall describe the rules of tracing any cartesian curve.

13.2. SECTION I : RULES FOR TRACING OF A CARTESIAN CURVE

(i) The curve passes through the origin :

Firstly we should see whether the curve passes through the origin or not. For this, we put $x = 0$, $y = 0$ in the equation of the given curve. If both the sides of the equation of the curve are satisfied then we say that the curve passes through the origin (0, 0).

For example, the curve $y^2 = 4ax$ (parabola) passes through the origin (0, 0).

(ii) Symmetry :

After this, we consider the symmetry of a given curve. The symmetry of a curve may occur in the following important ways.

(a) If the equation of the curve contains only even powers of y (so that the equation of the curve does not change by putting $-y$ for y in it), then the curve is said to be symmetrical about the x-axis.

When we say that the curve is symmetrical about the x-axis, it means that the portion of the curve above the x-axis will be exactly the same below the x-axis.

(b) If the equation of the curve contains only even powers of x, then the curve is said to be symmetrical about the y-axis.

(c) If the equation of the curve contains only even powers of x and y, then the curve is said to be symmetrical about both the axes;

e.g. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is symmetrical about both the axes.

(d) If the equation of the curve does not change by putting y for x and x for y in the given equation, then we say that the curve is symmetrical about the line $y = x$. For example, $x^3 + y^3 = 3axy$ is symmetrical about the line $y = x$.

(iii) Point of intersection of the curve with the co-ordinate axes :

Now we should see at what points the curve cuts the x-axis and y-axis. In order to determine the co-ordinates of the point of intersection with the x-axis we shall put $y = 0$ in the equation of the curve because the y-co-ordinate of any point situated on the x-axis is $=0$.

Similarly, in order to find out the co-ordinates of the point of intersection with the y-axis we shall put $x = 0$ in the equation of the curve.

NOTE : If the curve is symmetrical about the line $y = x$, then we should also find out the co-ordinates of the point of intersection of the curve with $y = x$.

(iv) Region or regions of the curve :

We shall solve for y from the equation of the given curve. If, for example, suppose that by giving to x, the values $x > a$ or $x < -a$, the values of y become imaginary, then it means that there would be no portion of the curve on the R.H.S. of $x = a$ or there would be no portion of the curve on the L.H.S. of $x = -a$.

NOTE : Sometimes it happens that the value of y increases corresponding to increasing values of x. In this case, the extent of the given curve will be up to infinity.

(v) Equation of tangent at the origin :

If the curve passes through the origin, then we should find out the equation of the tangent to the curve at the origin. In order to find out the equation of the tangent at the origin, the formula is to equate to zero the terms of the lowest degree from the equation of the curve. For example, in the curve $y^2 = 4ax$, the term of the lowest degree is $4ax$. Therefore $x = 0$ i.e. y-axis is the equation of the tangent at the origin to the curve (parabola) $y^2 = 4ax$.

(vi) Asymptotes, if any :

For those curves, which possess asymptotes, we should also find out their asymptotes by applying the appropriate rules.

To sum up, the aforesaid rules in general are observed for drawing the graph of a cartesian curve.

13.3. SECTION II : RULES FOR TRACING OF A POLAR CURVE

(i) The curve passes through the pole :

If $r = 0$ for any real value of θ; we say that the curve passes through the origin.

(ii) Symmetry :

(a) If by putting $-\theta$ for θ, the equation of the curve does not change, we say that the curve is symmetrical about the initial line OX.

(b) If by putting $\pi - \theta$ for θ, the equation of the curve does not change, we say that the curve is symmetrical about the line $\theta = \frac{\pi}{2}$ i.e. about the line OY.

(c) If by putting $-r$ for r, the equation of the curve does not change; i.e. if the equation contains only the even powers of r, we say that the curve is symmetrical about the pole.

(iii) Intersection of the curve with the initial line OX and also with OY.

For this, we find out the value of r by putting $\theta = 0$ and $\theta = \frac{\pi}{2}$ in the equation of the curve.

(iv) Region of the curve :

We also determine those regions of the curve which do not contain any portion of the curve and they are determined by those values of θ for which r is imaginary.

(v) Tangent at the pole :

If the curve passes through the origin, then we get the equation of the tangent at the origin corresponding to those values of θ for which $r = 0$.

(vi) Asymptotes, if any :

If the curve extends up to infinity, then we should also find out the asymptote of the curve from the formula in the polar form. Otherwise, it is convenient to determine the asymptote by changing the equation of the curve in cartesian form.

(vii) Some more points :

(a) We should plot some special points on the curve. For this, we take some special values of θ, say $\theta = 0, \theta = \frac{\pi}{6}, \theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{3}, \theta = \frac{\pi}{2}$ ... and after determining the corresponding values of r we plot these points on the curve.

(b) Sometimes it is convenient to trace out a curve by transforming the curve in the cartesian form.

(c) As per our requirement, we find out the angle ϕ between the radius vector and the tangent by using the formula $\tan\phi = \frac{r d\theta}{dr}$ so that we can determine the co-ordinates of those points where $\phi = \frac{\pi}{2}$ is any specified angle.

Now we use the aforesaid rules of curve-tracing in drawing the graph of some special curves.

13.4. SOME WELL-KNOWN CURVES

1. Curve $x^{2/3} + y^{2/3} = a^{2/3}$ (Astroid)

(i) Since by putting $x = 0$, $y = 0$, both the sides of the curve are not satisfied, hence the curve does not pass through the origin.

(ii) The equation of the given curve does not change when we put $-x$ for x and $-y$ for y in it and therefore the curve is symmetrical about both the axes.

(iii) Putting $y = 0$ in the equation of the curve, we get $x^{2/3} = a^{2/3}$ or $x^2 = a^2$ $\therefore$ $x = \pm a$ i.e. the curve cuts the x-axis at the point $(a, 0)$ and $(-a, 0)$. Similarly putting $x = 0$, we get $y^{2/3} = a^{2/3}$ $\therefore$ $y = \pm a$ i.e. the curve cuts the y-axis at the points $(0, a)$ and $(0, -a)$.

(iv) Again from the equation of the curve

$$y^{2/3} = a^{2/3} - x^{2/3}.$$

$\therefore$ If we put in this equation any quantity $x > a$ or $x < -a$, then $y^{2/3} = -ve$ $\therefore$ $y^2 = -ve$.

Therefore y becomes imaginary i.e., there is no portion of the curve on the R.H.S. of $x = +a$ or on the L.H.S. of $x = -a$. Similarly by solving for x from the given equation, it can be shown that there is no portion of the curve above $y = +a$ or below $y = -a$.

That is, the given curve is enclosed by four points $(a, 0)$, $(-a, 0)$, $(0, a)$ and $(0, -a)$.

Also, by comparing the curve $x^{2/3} + y^{2/3} = a^{2/3}$ with the circle $x^2 + y^2 = a^2$ we find that since $2/3 < 2$, the given curve will be inside the circle $x^2 + y^2 = a^2$.

Hence the graph of the curve will be like this :